

Semiparametric efficiency bounds for seemingly unrelated conditional moment restrictions

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Abstract

This paper addresses the problem of semiparametric efficiency bounds for conditional moment restriction models with different conditioning variables. We characterize such an efficiency bound, that in general is not explicit, as a limit of explicit efficiency bounds for a decreasing sequence of unconditional (marginal) moment restriction models. An iterative procedure for approximating the efficient score when this is not explicit is provided. Our theoretical results complete and extend existing results in the literature, provide new insight for the theory of semiparametric efficiency bounds literature and open the door to new applications. In particular, we investigate a class of regression-like (mean regression, quantile regression,...) models with missing data.

1 The model

Conditional moment restriction models represent a large class of statistical models. Seemingly unrelated nonlinear regressions, see Gallant (1975), Müller (2009), seemingly unrelated quantile regressions, see Jun and Pinske (2009), regression models with missing data, see Robins, Rotnitzky and Zhao (1994), Tsiatis (2006), are only few examples and related contributions. Ai and Chen (2009) and Hansen (2007) provide many other references and examples of econometric models that could be stated as conditional moment restriction models.

In this paper we address the problem of calculating semiparametric efficiency bounds in models defined by several conditional moment restrictions with possibly different conditioning variables. More formally, the sample under study consists of independent copies

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of a random vector $Z \in \mathcal{Z} \subset \mathbb{R}^q$. Let J be some positive integer that is fixed in the following. For any $j \in \{1, \dots, J\}$, let $X^{(j)}$ be a random q_j -dimension subvector of Z , where $0 \leq q_j < q$. Let $g_j : \mathbb{R}^q \times \mathbb{R}^d \rightarrow \mathbb{R}^{p_j}$, $j \in \{1, \dots, J\}$, denote given functions of Z and the unknown parameter $\theta \in \Theta \subset \mathbb{R}^d$. The semiparametric model we consider is defined by the conditional moment restrictions

$$E[g_j(Z, \theta) \mid X^{(j)}] = 0, \quad j = 1, \dots, J, \quad \text{almost surely.} \quad (1)$$

It is assumed that the d -dimension parameter θ is identified by the conditional restrictions, which means there exists a unique value θ_0 such that the true law of Z satisfies equations (1). By definition, $X^{(j)}$ is a constant random variable when $q_j = 0$, and hence the conditional expectation given $X^{(j)}$ is the marginal expectation.

Particular cases of this model have been extensively studied in the literature. For $J = 1$ and $q_1 = 0$ we obtain a model defined by an unconditional set of moment equations

$$E[g(Z, \theta)] = 0.$$

Hansen (1982) considered the class of GMM estimators and showed how to construct an optimal one in this class. Its asymptotic variance equals the the semiparametric efficiency bound obtained by Chamberlain (1987).

The GMM method extends naturally to models defined by conditional moment equations, corresponding to the case $J = 1$ and $q_1 > 0$ in our setting, that is

$$E[g(Z, \theta) \mid X] = 0.$$

From a mathematical point of view, such a model is equivalent to the intersection of the models of the form

$$E[a(X) g(Z, \theta)] = 0,$$

where $a(X)$ is an arbitrary conformable random matrix whose entries are square integrable. Following the econometric literature, $a(X)$ is referred to as a matrix of *instruments*. The supremum of the information on θ_0 in these models yields the semiparametric Fisher information on θ_0 in the conditional equation model, obtained by Chamberlain (1992a). It is also the information on θ_0 for the unconditional moment equation

$$E[a^*(X) g(Z, \theta)] = 0,$$

with properly chosen ‘optimal’ instruments $a^*(X)$.

A further generalization, which can also be written under the form (1), is given by a sequential (nested) moment restrictions model, in which the σ -fields generated by the conditioning vectors satisfy the condition $\sigma(X^{(1)}) \subset \sigma(X^{(2)}) \subset \dots \subset \sigma(X^{(J)})$. For the expression of the semiparametric efficiency bound in the sequential case, see Chamberlain (1992b) and Ai and Chen (2009); see also Hahn (1997) and Ahn and Schmidt (1999) and references therein for examples of applications. It turns out that once again the

information on θ_0 can be obtained by taking the supremum of the information on θ_0 in the following unconditional models :

$$E \left[a_j \left(X^{(j)} \right) g_j (Z, \theta) \right] = 0, \quad j = 1, \dots, J,$$

where the number of lines of the matrices a_j is fixed and equal to the dimension of θ and the supremum is attained for a suitable choice $a_1^* \left(X^{(1)} \right), \dots, a_J^* \left(X^{(J)} \right)$ of optimal instruments. The reason why this happens in the case with nested σ -fields is the fact that the model of interest can be written as the decreasing limit of a sequence of models for which a so-called ‘*spanning condition*’, similar to the one considered in Newey (2004), holds and the limit of the corresponding efficient scores has an explicit solution.

In this paper we show that the information on θ_0 in model (1) can be obtained as the limit of the information on θ_0 in a decreasing sequence of unconditional moment models of the form

$$E \left[a_j^{(k)} \left(X^{(j)} \right) g_j (Z, \theta) \right] = 0, \quad j = 1, \dots, J, \quad k = 1, 2, \dots \quad (2)$$

where the numbers of lines in the matrices $a_j^{(k)}$ increases to infinity with k . To our best knowledge this result is new. It provides theoretical support for a natural solution that could be used in practice: replace the model (1) by a large number of unconditional moment conditions like (2) in order to approach efficiency. Herein we also propose an alternative route for approximating the efficiency bound. More precisely, we give a general method to approximate the efficient score, which in most of the situations does not have an explicit form as in the aforementioned examples. In particular, our general approach for approximating the efficient score brings in a new light the functional equations used to characterize the efficient score in the regression model with unobserved explanatory variables in Robins, Rotnitzky and Zhao (1994); see also Tsiatis (2006) and Tan (2011). To summarize, our theoretical results complete and extend existing results in the literature, provide new insight for the theory of semiparametric efficiency bounds literature and open the door to new applications, in particular in missing data contexts.

The paper is organized as follows. Section 2 contains our main results. We show that under a suitable ‘spanning condition’ on the tangent spaces, the semiparametric Fisher information in model (1) can be obtained as the limit of the efficiency bounds for a decreasing sequence of models. In section 3 we propose a ‘backfitting’ procedure, for computing the projection of the score on the tangent space of the model. With at hand an approximation of the efficient score, we suggest a general method for constructing asymptotically efficient estimators. In section 4 we illustrate the utility of our theoretical results for two large classes of models: sequential (nested) conditional models and regression-like models with missing data. The technical assumptions required for our results and some technical proofs are relegated to the Appendix.

2 The main results

Let us introduce some notation and definitions, see also van der Vaart (1998), sections 25.2 and 25.3. Given a sample space \mathcal{Z} and a probability P on the sample space, we denote by $L^2(P)$ the usual Hilbert space of measurable real-valued functions that are squared-integrable with respect to P . For \mathcal{H} a Hilbert space and $\mathcal{S} \subset \mathcal{H}$ let $\overline{\mathcal{S}}$ denote the closure of \mathcal{S} in \mathcal{H} . Moreover, if $\mathcal{S} \subset \mathcal{H}$ is a linear subspace and $h \in \mathcal{H}$, let $\Pi(h|\mathcal{S})$ be the projection of h on $\overline{\mathcal{S}}$. The statistical models on the sample space \mathcal{Z} , are denoted by $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \dots$. A statistical model is a collection of probability measures defined by their densities with respect to some fixed dominating measure on the sample space. For a model \mathcal{P} (resp. \mathcal{P}_j) and a probability measure P in the model, let $\dot{\mathcal{P}}_P$ (resp. $\dot{\mathcal{P}}_{j,P}$) denote the tangent cone of the model \mathcal{P} (resp. \mathcal{P}_j) at P . When there is no possible confusion, we simply write $\dot{\mathcal{P}}_P$ (resp. $\dot{\mathcal{P}}_{j,P}$). Let $\mathcal{T}(\mathcal{P}, P)$ denote the tangent space of a model \mathcal{P} at some probability measure $P \in \mathcal{P}$, that means the closure of the linear span of the tangent set $\dot{\mathcal{P}}_P$. By definition, both the tangent cone and the tangent space are subsets of $L^2(P)$. Herein the vectors are column matrices and $A \in \mathbb{R}^r \times \mathbb{R}^s$ means A is a $r \times s$ -matrix with random elements, if not stated differently. For $A \in \mathbb{R}^r \times \mathbb{R}^r$, $E(A)$ denotes the expectation of A and $E^{-1}(A)$ denotes the inverse of the square matrix $E(A)$. Finally, for a square matrix A , let A^- denote a generalized inverse, for instance the Moore-Penrose pseudoinverse.

2.1 A general lemma

The following result is a generalization of Theorem 1 in Newey (2004) where only the case of conditioning vectors $X^{(j)}, j = 1, \dots, J$, that generate the same σ -field is considered. The proof of our result is postponed to the Appendix.

Lemma 1 *Let $P_0 \in \mathcal{P} \subset \mathcal{P}_1$ be the true law of the vector $Z \in \mathcal{Z}$ and $\theta_0 = \psi(P_0)$ for a map $\psi : \mathcal{P}_1 \rightarrow \mathbb{R}^d$ differentiable at P_0 relative to the tangent cone $\dot{\mathcal{P}}_{1,P_0}$. Let $\{\mathcal{P}_k\}_{k \in \mathbb{N}^*}$ be a decreasing family of statistical models such that*

$$\mathcal{P}_1 \supset \mathcal{P}_2 \supset \dots \supset \mathcal{P}_j \supset \mathcal{P}_{k+1} \supset \dots \supset \bigcap_{k=1}^{\infty} \mathcal{P}_k \supset \mathcal{P} \ni P_0 \quad (3)$$

and

$$\bigcap_{k=1}^{\infty} \mathcal{T}_k = \mathcal{T}, \quad (4)$$

where $\mathcal{T} = \mathcal{T}(\mathcal{P}, P_0)$ and $\mathcal{T}_k = \mathcal{T}(\mathcal{P}_k, P_0)$, $k \in \mathbb{N}^*$. Then

$$I_{\theta_0}(\mathcal{P}) = \lim_{k \rightarrow \infty} I_{\theta_0}(\mathcal{P}_k),$$

where $I_{\theta_0}(\mathcal{P})$ stands for the Fisher information on $\theta_0 = \psi(P_0)$ in the model \mathcal{P} .

For the definition of the Fisher information $I_{\theta_0}(\mathcal{P})$ on $\theta_0 = \psi(P_0)$ in the model \mathcal{P} we refer to Bickel, Klaassen, Ritov and Wellner (1993) or van der Vaart (1998); see also Newey (1990). When the models \mathcal{P}_k , $k \in \mathbb{N}^*$, are defined by an increasing number of moment conditions with the same conditioning vectors, condition (4) is exactly the so-called spanning condition of Newey (2004).

Remark 1 *Even if $\bigcap_{k=1}^{\infty} \mathcal{P}_k = \mathcal{P}$, condition (4) is not necessarily fulfilled. To see this, consider a symmetric density f_0 on the real line and let s_1, s_2 be two odd functions such that $|s_1|, |s_2| \leq 1$ (e.g. $s_l(x) = x^{2l-1} \mathbf{I}_{\{|x| \leq 1\}}$, $l = 1, 2$). For any $k \in \mathbb{N}^*$ and $t \in [-1, 1]$, define*

$$f_t(x) = f_0(x)[1 + t s_2(x)], \quad f_{t;k}(x) = k f_0(kx)[1 + t s_1(x)]$$

and consider the following models defined by their densities with respect to $\lambda_{\mathbb{R}}$ the Lebesgue measure on the real line : $\mathcal{Q}_k = \{f_{t;k} \cdot \lambda_{\mathbb{R}} : t \in [-1, 1]\}$, $k \in \mathbb{N}^$, and*

$$\mathcal{P} = \{f_t \cdot \lambda_{\mathbb{R}} : t \in [-1, 1]\}, \quad \mathcal{P}_k = \mathcal{P} \cup \bigcup_{m=k}^{\infty} \mathcal{Q}_m, \quad k \in \mathbb{N}^*.$$

Then we have

$$\mathcal{P}_1 \supset \mathcal{P}_2 \supset \dots \supset \mathcal{P}_k \supset \mathcal{P}_{k+1} \supset \dots \supset \bigcap_{k=1}^{\infty} \mathcal{P}_k = \mathcal{P}.$$

To describe the corresponding tangent spaces, notice that

$$\forall k \geq 1, \quad \partial_t \log f_{t;k}(x)|_{t=0} = s_1(x) \quad \text{and} \quad \partial_t \log f_t(x)|_{t=0} = s_2(x),$$

and thus $\dot{\mathcal{P}} = \{a s_2(x) : a \in \mathbb{R}\}$,

$$\dot{\mathcal{P}}_k = \{a s_2(x) : a \in \mathbb{R}\} \cup \{b s_1(x) : b \in \mathbb{R}\}, \quad k \in \mathbb{N}^*.$$

Then $\mathcal{T} = \{a s_2(x) : a \in \mathbb{R}\}$,

$$\mathcal{T}_k = \{a s_2(x) + b s_1(x) : a, b \in \mathbb{R}\}, \quad k \in \mathbb{N}^*.$$

This shows that

$$\bigcap_{k=1}^{\infty} \dot{\mathcal{P}}_k \not\supseteq \dot{\mathcal{P}} \quad \text{and} \quad \bigcap_{k=1}^{\infty} \mathcal{T}_k \not\supseteq \mathcal{T},$$

even if the decreasing sequence of models $\{\mathcal{P}_k\}_{k \in \mathbb{N}^}$ is such that $\bigcap_{k=1}^{\infty} \mathcal{P}_k = \mathcal{P}$.*

2.2 Efficiency bound

The main idea we follow to derive the semiparametric efficiency bound for the parameter θ_0 is to transform the finite number of conditional moment restrictions (1) in a countable number of unconditional (marginal) moment restrictions. Next, for any finite subset of these unconditional moment restrictions, one could easily obtain the Fisher information bound. Eventually, one may expect to obtain the semiparametric efficiency bound for the model (1) as the limit of the efficiency bounds for a decreasing sequence of models defined by an increasing sequence of finite subsets of unconditional moment restrictions. Remark 1 proves that in general this intuition is not correct. However, Lemma 1 states that this intuition becomes correct under the additional condition (4).

Let us introduce some more notation. If $\zeta : \mathcal{Z} \times \Theta \rightarrow \mathbb{R}^m$, $m \geq 1$, is some given function of Z and θ and X is some subvector of Z , we denote

$$E[\partial_{\theta'} \zeta \mid X] = E[\partial_{\theta'} \zeta(Z, \theta_0) \mid X] = \left. \frac{\partial}{\partial \theta'} E[\zeta(Z, \theta_0) \mid X] \right|_{\theta=\theta_0} \in \mathbb{R}^d \times \mathbb{R}^m, \quad (5)$$

when such derivatives of $\theta \mapsto E[\zeta(Z, \theta) \mid X]$ exist. A similar notation will be used with the conditional expectation $E(\cdot \mid X)$ replaced by the marginal (unconditional) expectation with respect to the law of Z . Let us point out that the maps $\theta \mapsto \zeta(z, \theta)$ may not be everywhere differentiable. Next, let us define

$$\underline{g} = (g'_1, \dots, g'_J)' \in \mathbb{R}^p = \mathbb{R}^{p_1 + \dots + p_J},$$

and let \underline{X} denote the vector of all components of Z contained in the subvectors $X^{(j)}$, $j = 1, \dots, J$.

For the purpose of transforming conditional moments in unconditional versions, consider a countable set of squared integrable functions $\mathcal{W} = \{w_k : k \in \mathbb{N}^*\} \subset L^2(P_0)$ such that $\overline{\text{lin}} \mathcal{W} = L^2(P_0)$, that is the linear span of \mathcal{W} is dense in $L^2(P_0)$. For any $s \in \mathbb{N}^*$, define a $p \times p$ -diagonal matrix

$$\begin{aligned} \underline{w}_s(\underline{X}) = & \text{diag}(\underbrace{E[w_s(Z) \mid X^{(1)}], \dots, E[w_s(Z) \mid X^{(1)}]}_{p_1}, \dots \\ & \dots, \underbrace{E[w_s(Z) \mid X^{(J)}], \dots, E[w_s(Z) \mid X^{(J)}]}_{p_J}). \end{aligned}$$

Next, for any $k \in \mathbb{N}^*$, let

$$\underline{w}^{(k)}(\underline{X}) = (\underline{w}_1(\underline{X}), \dots, \underline{w}_k(\underline{X}))' \in \mathbb{R}^{kp} \times \mathbb{R}^p \quad \text{and} \quad \underline{g}_k^w(Z, \theta) = \underline{w}^{(k)}(\underline{X}) \underline{g}(Z, \theta) \in \mathbb{R}^{kp}.$$

Moreover, let $I_{\theta_0}^{(k)}$ be the Fisher information on θ_0 in the model

$$E[\underline{g}_k^w(Z, \theta)] = 0, \quad (6)$$

that is

$$I_{\theta_0}^{(k)} = E \left[\left(\partial_{\theta'} g_k^w(Z, \theta_0) \right)' \right] V^{-} \left[g_k^w(Z, \theta_0) \right] E \left[\partial_{\theta'} g_k^w(Z, \theta_0) \right].$$

See Chamberlain (1987), Newey (2001), see also Chen and Pouzo (2009) for the non-smooth case.

We can state now the main result of the paper.

Theorem 1 *Under the Assumptions T and SP in the Appendix, the information bound I_{θ_0} on θ_0 at P_0 in model (1) is given by*

$$I_{\theta_0} = \lim_{k \rightarrow \infty} I_{\theta_0}^{(k)},$$

where, for any $k \in \mathbb{N}^*$, $I_{\theta_0}^{(k)}$ is the Fisher information on θ_0 in the model defined as in (6).

Proof. For any $k \in \mathbb{N}^*$, let \mathcal{P}_k be the model defined by equation (6) and \mathcal{P} the model defined by equation (1). Then

$$\mathcal{P}_1 \supset \mathcal{P}_2 \supset \dots \supset \mathcal{P}_k \supset \mathcal{P}_{k+1} \supset \dots \supset \bigcap_{k=1}^{\infty} \mathcal{P}_k = \mathcal{P}.$$

Hence the stated result is a direct consequence of Lemma 1, provided that condition (4) holds for the tangent spaces of \mathcal{P} and \mathcal{P}_k , $k \in \mathbb{N}^*$, at θ_0 .

For each $j \in \{1, \dots, J\}$, any $z \in \mathcal{Z} \subset \mathbb{R}^q$ could be partitioned in two subvectors $y^{(j)} \in \mathbb{R}^{q-q_j}$ and $x^{(j)} \in \mathbb{R}^{q_j}$ with $x^{(j)}$ in the support of $X^{(j)}$. Let $P_{X^{(j)}}$ denote the law of $X^{(j)}$. Model \mathcal{P} is then defined by the set of conditions

$$\int g_j(z, \theta) f(z, \theta) dy^{(j)} = 0 \quad P_{X^{(j)}} - a.s., \quad j \in \{1, \dots, J\}; \quad (7)$$

for a fixed k , the model \mathcal{P}_k is defined by

$$\int g_j(z, \theta) f(z, \theta) \bar{w}_s^j(x^{(j)}) dz = 0, \quad j \in \{1, \dots, J\}, \quad s \in \{1, \dots, k\}, \quad (8)$$

where

$$\bar{w}_s^j(x^{(j)}) = E[w_s(Z) | X^{(j)} = x^{(j)}].$$

Consider now a regular parametric family $\{f_t\}_{t \in (-\varepsilon, \varepsilon)}$ of densities satisfying (7), that means that there exist parameters $\theta_t \in \Theta$, such that, for any $t \in (-\varepsilon, \varepsilon)$ and $P_{X^{(j)}} - a.s.$,

$$\int g_j(z, \theta_t) f_t(z, \theta_t) dy^{(j)} = 0, \quad \forall j \in \{1, \dots, J\}. \quad (9)$$

Let

$$\begin{aligned}\dot{\theta} &= \left. \frac{\partial \theta_t}{\partial t} \right|_{t=0}, \\ s &= \left. \partial_t \log f_t(Z, \theta_t) \right|_{t=0}, \quad S_{\theta_0} = \left. \partial_\theta \log f(Z, \theta) \right|_{\theta=\theta_0}, \\ s_1 &= \left. \partial_t \log f_t(Z, \theta_t) \right|_{t=0} = s + S'_{\theta_0} \dot{\theta}.\end{aligned}$$

Here and in the following, the derivatives of the log-densities are to be understood in the mean square sense, see Ibragimov and Has'minskii (1981), page 64. Differentiating with respect to t in (9) we obtain

$$E \left[\partial_{\theta'} g_j(Z, \theta_0) | X^{(j)} \right] \dot{\theta} + E \left[g_j(Z, \theta_0) s_1(Z) | X^{(j)} \right] = 0, \quad \forall j \in \{1, \dots, J\}. \quad (10)$$

Since $\dot{\theta} \in \mathbb{R}^d$ could be arbitrary, we deduce that for each $j \in \{1, \dots, J\}$,

$$E \left[\partial_{\theta'} g_j(Z, \theta_0) | X^{(j)} \right] + E \left[g_j(Z, \theta_0) S'_{\theta_0}(Z) | X^{(j)} \right] = 0, \quad E \left[g_j(Z, \theta_0) s(Z) | X^{(j)} \right] = 0.$$

The last equation and the expression of the score functions s_1 suggest a tangent space $\mathcal{T} = \mathcal{T}(\mathcal{P}, P_0)$ of the form

$$\mathcal{T} = \overline{\text{lin}} S_{\theta_0} + \left\{ s : E(s^2) < \infty, E(s) = 0, E \left[g_j(Z, \theta_0) s(Z) | X^{(j)} \right] = 0, 1 \leq j \leq J \right\}. \quad (11)$$

On the other hand, the tangent space $\mathcal{T}_k = \mathcal{T}(\mathcal{P}_k, P_0)$ corresponding to the model defined by the equations (8) is given by vectors satisfying the unconditional moment equations

$$E \left[\partial_{\theta'} g_j(Z, \theta_0) \overline{w}_r^j(X^{(j)}) \right] \dot{\theta} + E \left[g_j(Z, \theta_0) s_1(Z) \overline{w}_r^j(X^{(j)}) \right] = 0, \quad (12)$$

$1 \leq j \leq J, 1 \leq r \leq k$. This yields the tangent spaces

$$\begin{aligned}\mathcal{T}_k &= \overline{\text{lin}} S_{\theta_0} + \left\{ s : E(s^2) < \infty, E(s) = 0, E \left[g_j(Z, \theta_0) s(Z) \overline{w}_r^j(X^{(j)}) \right] = 0, \right. \\ &\quad \left. \forall 1 \leq j \leq J, \forall 1 \leq r \leq k \right\};\end{aligned}$$

see for instance Example 3, section 3.2 in Bickel, Klaassen, Ritov and Wellner (1993). Since the functions $w_k(Z), k \in \mathbb{N}^*$, span $L^2(P_0)$, their projections $\overline{w}_k^j(X^{(j)})$ on $L^2(P_{X^{(j)}})$, $k \in \mathbb{N}^*$, will span $L^2(P_{X^{(j)}})$. Consequently, equations (10) are satisfied if and only if equations (12) are satisfied for any $k \in \mathbb{N}^*$. In other words, the equivalent of the spanning condition of Newey (2004), see our equation (4) above, is satisfied and we can apply Lemma 1 to conclude that $I_{\theta_0} = \lim_{k \rightarrow \infty} I_{\theta_0}^{(k)}$.

The proof will be complete if we show that the tangent space $\mathcal{T} = \mathcal{T}(\mathcal{P}, P_0)$ is indeed the set described in equation (11). Consider for simplicity that $J = 2$, the general case could be handled similarly. It is quite easy to see that equations (10) guarantees the

inclusion “ \subset ” in display (11). To show the reverse inclusion, it suffices to prove that $\mathcal{T}' \subset \mathcal{T}$, where

$$\begin{aligned} \mathcal{T}' = \mathcal{T}'(\mathcal{P}, P_0) = \{s : E(s^2) < \infty, E(s) = 0, \\ E[g_1(Z, \theta_0) s(Z) | X^{(1)}] = 0, E[g_2(Z, \theta_0) s(Z) | X^{(2)}] = 0\}. \end{aligned}$$

Let f_0 denote the true density of the vector Z . Take $s \in \mathcal{T}'$ and suppose for the moment that s is bounded. Then, for real numbers t with sufficiently small absolute values, the functions $f_t = (1 + t \cdot s) f_0$ are densities on \mathcal{Z} and if E_{f_t} denotes expectation with respect to the law defined by f_t ,

$$E_{f_t}[g_j(Z, \theta_0) a(X^{(j)})] = E[g_j(Z, \theta_0) a(X^{(j)})] + t E[g_j(Z, \theta_0) s(Z) a(X^{(j)})] = 0,$$

for any square-integrable function $a(X^{(j)})$, so that $E_{f_t}[g_j(Z, \theta_0) | X^{(j)}] = 0$, $j = 1, 2$. Moreover,

$$\partial_t \log f_t|_{t=0} = \partial_t \log(1 + t \cdot s)|_{t=0} = s,$$

which means that the family of densities $\{f_t\}_{|t| < \varepsilon}$ defines a submodel of model (1) for which the tangent vector at $t = 0$ is exactly s . Next, we have to extend the argument to unbounded functions s . If $\mathcal{M} \subset L^2(P_0)$ is the subspace of bounded functions of Z , it remains to show that $\mathcal{M} \cap \mathcal{T}'$ is dense in \mathcal{T}' . One may consider this step obvious since any unbounded square integrable function can be approximated by a sequence of bounded functions, see for instance Ai and Chen (2003), page 1838. We argue that this well-known approximation result cannot be directly applied to our context, as it is also the case in other contexts considered in the efficiency bounds literature. Indeed, here we are in the following situation: we have two infinite-dimension closed subspaces \mathcal{T}'_1 and \mathcal{T}'_2 such that $\mathcal{T}' = \mathcal{T}'_1 \cap \mathcal{T}'_2$, $\overline{\mathcal{M} \cap \mathcal{T}'_1} = \mathcal{T}'_1$ and $\overline{\mathcal{M} \cap \mathcal{T}'_2} = \mathcal{T}'_2$, and we need that $\overline{\mathcal{M} \cap \mathcal{T}'} = \mathcal{T}'$. To our best knowledge, there is no general mathematical result which would allow us to claim that $\mathcal{M} \cap \mathcal{T}'$ is dense in \mathcal{T}' without any further argument. That is why we have to provide a proof adapted to the case we consider herein. By Assumption T and the subsequent remark, and equation (28), there exist two bounded vector functions b_1 and b_2 defined like in equation (28) such that, for $i, j \in \{1, 2\}$, $i \neq j$,

$$E(g_i b'_i | X^{(i)}) = 0 \quad \text{and} \quad \|E^{-1}(g_i b'_j | X^{(1)}, X^{(2)})\|_\infty < 1,$$

where $g_i = g_i(Z, \theta_0)$. Here and in the sequel, the norm of a vector (or matrix) should be understood as the sum of componentwise norms. Since \mathcal{M} is dense in $L^2(P_0)$, for a fixed $s \in \mathcal{T}'$ there exist a sequence $\{t_n\}_n \subset \mathcal{M}$ such that

$$\|s - t_n\|_{L^2(P_0)} \xrightarrow{n \rightarrow \infty} 0.$$

Define

$$u_n = t_n - E(t_n g'_1 | X^{(1)}) E^{-1}(b_1 g'_1 | X^{(1)}) b_1 - E(t_n g'_2 | X^{(2)}) E^{-1}(b_2 g'_2 | X^{(2)}) b_2.$$

It is clear that we can take $\{t_n\}_n \subset \mathcal{M}$ such that

$$\|E(t_n g_1 | X^{(1)})\|_\infty + \|E(t_n g_2 | X^{(2)})\|_\infty < \infty$$

and thus $u_n \in \mathcal{M}$. Then

$$\begin{aligned} E(g_1 u'_n | X^{(1)}) &= \frac{E(g_1 t'_n | X^{(1)}) - E(g_1 b'_1 | X^{(1)}) E^{-1}(g_1 b'_1 | X^{(1)}) E(g_1 t'_n | X^{(1)})}{-E[g_1 b'_2 E^{-1}(g_2 b'_2 | X^{(2)}) E(g_2 t'_n | X^{(2)}) | X^{(1)}]} \\ &= -E[\underbrace{E(g_1 b'_2 | X^{(1)}, X^{(2)})}_{=0} E^{-1}(g_2 b'_2 | X^{(2)}) E(g_2 t'_n | X^{(2)}) | X^{(1)}] \\ &= 0, \end{aligned} \tag{13}$$

and similarly,

$$E(g_2 u'_n | X^{(2)}) = 0. \tag{14}$$

Moreover,

$$\begin{aligned} s - u_n &= s - t_n + t_n - u_n \\ &= s - t_n + E[(t_n - s) g'_1 | X^{(1)}] E^{-1}(b_1 g'_1 | X^{(1)}) b_1 \\ &\quad + E[(t_n - s) g'_2 | X^{(2)}] E^{-1}(b_2 g'_2 | X^{(2)}) b_2, \end{aligned}$$

which entails

$$\begin{aligned} \|s - u_n\|_{L^2(P_0)} &\leq \|s - t_n\|_{L^2(P_0)} + \|E[(t_n - s) g'_1 | X^{(1)}]\|_{L^2(P_0)} \cdot \|b_1\|_\infty \\ &\quad + \|E[(t_n - s) g'_2 | X^{(2)}]\|_{L^2(P_0)} \cdot \|b_2\|_\infty. \end{aligned}$$

Noting that

$$\begin{aligned} \|E[(t_n - s) g'_1 | X^{(1)}]\|_{L^2(P_0)}^2 &= E\{E^2[(t_n - s) g'_1 | X^{(1)}]\} \\ (Cauchy - Schwarz) &\leq E\{E^2[(t_n - s) | X^{(1)}] E^2(g'_1 | X^{(1)})\} \\ &\leq \|E(g_1 | X^{(1)})\|_\infty^2 E\{E^2[(t_n - s) | X^{(1)}]\} \\ (Jensen) &\leq \|E(g_1 | X^{(1)})\|_\infty^2 E\{E[(t_n - s)^2 | X^{(1)}]\} \\ &\leq \|E(g_1 | X^{(1)})\|_\infty^2 \|t_n - s\|_{L^2(P_0)}^2, \end{aligned}$$

we finally obtain $\|s - u_n\|_{L^2(P_0)} \rightarrow 0$ as $n \rightarrow \infty$. In particular, deduce that $E(u_n) \rightarrow 0$. Now, since all the previous equations and inequalities involving u_n hold also with u_n replaced by $u_n - E(u_n)$, deduce that $\{u_n - E(u_n)\}_n \subset \mathcal{M} \cap \mathcal{T}'$, which implies that $s \in \overline{\mathcal{M} \cap \mathcal{T}'}$. Now the proof is complete. ■

In the general theory of efficiency bounds, the semiparametric Fisher information on a finite dimension parameter in a semiparametric model is the infimum of the Fisher

information over all its parametric submodels; see for instance Newey (1990). For models defined by conditional moment equations, Theorem 1 shows that the same semiparametric Fisher information can be alternatively obtained as the lower limit of the semiparametric Fisher information in a sequence of decreasing supra-models. The main reason for this is that with such decreasing sequence of supra-models, the ‘spanning condition’ (4) holds true. Moreover, since $L^2(P_0)$ is a separable Hilbert space, Theorem 1 can be restated under the following equivalent form.

Corollary 1 *Under the conditions of Theorem 1,*

$$I_{\theta_0} = \sup_{b \in \mathcal{B}} I_{\theta_0}(b),$$

where

$$\mathcal{B} = \{ (b_1(X^{(1)}), \dots, b_J(X^{(J)})) : b_{j,lk} \in L^2(P_{X^{(j)}}) \ 1 \leq l \leq d, 1 \leq k \leq p_j, 1 \leq j \leq J \},$$

so that any $b = b(\underline{X}) \in \mathcal{B}$ is a $d \times p$ -matrix with random elements, and $I_{\theta_0}(b)$ is the Fisher information on θ_0 in the model defined by the marginal moment restrictions

$$E[b_j(X^{(j)}) g_j(Z, \theta)] = 0, \quad j \in \{1, \dots, J\}, \quad (15)$$

model which can also be written under the compact form $E[b(\underline{X}) \underline{g}(Z, \theta)] = 0$.

Remark 2 *We argue that, under further assumptions, the result of Theorem 1 extends to the case where the unknown functions g_j depend also on a same unknown function h of the observations and the parameter. More precisely, when the model is defined by*

$$E[\tilde{g}_j(Z, \theta, h(Z, \theta)) \mid X^{(j)}] = 0, \quad j = 1, \dots, J, \quad (16)$$

where $\tilde{g}_j : \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{p_h} \rightarrow \mathbb{R}^{p_j}$, $j \in \{1, \dots, J\}$, are known. With the same notations used for defining \underline{g}_k^w , let

$$\tilde{\underline{g}}_k^w(Z, \theta, h(Z, \theta)) = \underline{w}^{(k)}(\underline{X}) \tilde{\underline{g}}(Z, \theta, h(Z, \theta)) \in \mathbb{R}^{k_p}, \quad \forall k \in \mathbb{N}^*,$$

where $\tilde{\underline{g}} = (\tilde{g}_1', \dots, \tilde{g}_J')'$ and let $\tilde{I}_{\theta_0}^{(k)}$ be the Fisher information on θ_0 in the model

$$E[\tilde{\underline{g}}_k^w(Z, \theta, h(Z, \theta))] = 0; \quad (17)$$

its expression as a solution of a variational problem can be found in Chamberlain (1992), Ai and Chen (2003) or Chen and Pouzo (2009).

Similar but more involved arguments can be invoked to show the following result, which we state here as a conjecture: the information \tilde{I}_{θ_0} on θ_0 at P_0 in model (16) is given by

$$\tilde{I}_{\theta_0} = \lim_{k \rightarrow \infty} \tilde{I}_{\theta_0}^{(k)},$$

where $\tilde{I}_{\theta_0}^{(k)}$ is the Fisher information on θ_0 in model (17).

3 Efficient estimation

To simplify the presentation, let us take $J = 2$. To obtain an efficient estimator, a common way is to solve θ from the efficient score equations; see van der Vaart (1998), section 25.8. By definition, the efficient score is the componentwise projection of the score S_{θ_0} on the orthogonal complement of the tangent space $\mathcal{T} = \mathcal{T}(\mathcal{P}, P_0)$ defined in equation (11). In the projection of S_{θ_0} on \mathcal{T}^\perp only the nonparametric part of the tangent space matters. Moreover, the projection of S_{θ_0} is componentwise. It is then common practice in the literature to identify \mathcal{T} with the subspace of $\{L^2(P_0)\}^d = \bigoplus_{k=1}^d L^2(P_0)$ obtained as the d -fold cartesian product of the nonparametric part of \mathcal{T} . Here the direct sum of Hilbert spaces is considered with the usual inner product $\langle (\phi_1, \dots, \phi_d), (\psi_1, \dots, \psi_d) \rangle = \langle \phi_1, \psi_1 \rangle + \dots + \langle \phi_d, \psi_d \rangle$. Therefore we will slightly change our notation for the tangent spaces. More precisely, let us define

$$\begin{aligned} \mathcal{T} &= \left\{ s \in \bigoplus_{k=1}^d L^2(P_0) : E(s) = 0, E(g_i(Z, \theta_0)s'(Z) \mid X^{(i)}) = 0, i = 1, 2 \right\} \\ &= \mathcal{T}_1 \cap \mathcal{T}_2, \end{aligned}$$

where, for $i = 1, 2$,

$$\mathcal{T}_i = \left\{ s \in \bigoplus_{k=1}^d L^2(P_0) : E(s) = 0, E(g_i(Z, \theta_0)s'(Z) \mid X^{(i)}) = 0 \right\},$$

so that

$$\mathcal{T}_i^\perp = \left\{ s \in \bigoplus_{k=1}^d L^2(P_0) : s(Z) = a_i(X^{(i)}) g_i(Z, \theta_0) \right\}.$$

Clearly, $\mathcal{T}^\perp = \overline{\mathcal{T}_1^\perp + \mathcal{T}_2^\perp}$.

In general, the projection of S_{θ_0} on \mathcal{T}^\perp is not explicit. To approximate this projection and to further build an asymptotically efficient estimator for model (1), we use the iterative (“backfitting” or successive approximation) procedure considered in Theorem A.4.2 of Bickel, Klaassen, Ritov and Wellner (1993), page 438; BKRW hereafter. Let $H_i = \mathcal{T}_i^\perp$, $g_i = g(Z, \theta_0)$, $i = 1, 2$, and let $E(\partial_\theta g'_i)$ be the transposed of the matrix $E(\partial_{\theta'} g_i)$ defined in equation (5). The steps of the procedure we propose are the following :

1. Set $m = 0$. Take $a_1^{(0)} = 0$.
2. Put $m = m + 1$. Calculate

$$\overline{S}_{\theta_0}^{(m)} = a_1^{(m)}(X^{(1)}) g_1 + a_2^{(m)}(X^{(2)}) g_2$$

where

$$\begin{aligned}
a_1^{(m)}(X^{(1)}) &= a_1^{(m)}(X^{(1)}, \theta_0) = -E(\partial_\theta g'_1 | X^{(1)}) V^-(g_1 | X^{(1)}) \\
&+ E[E(\partial_\theta g'_2 | X^{(2)}) V^-(g_2 | X^{(2)}) g_2 g'_1 | X^{(1)}] V^-(g_1 | X^{(1)}) \\
&+ E[E[a_1^{(m-1)}(X^{(1)}) g_1 g'_2 | X^{(2)}] V^-(g_2 | X^{(2)}) g_2 g'_1 | X^{(1)}] V^-(g_1 | X^{(1)})
\end{aligned}$$

and

$$\begin{aligned}
a_2^{(m)}(X^{(2)}) &= a_2^{(m)}(X^{(2)}, \theta_0) = -E(\partial_\theta g'_2 | X^{(2)}) V^-(g_2 | X^{(2)}) \\
&- E[a_1^{(m)}(X^{(1)}) g_1 g'_2 | X^{(2)}] V^-(g_2 | X^{(2)}).
\end{aligned}$$

3. Repeat from step 2 till the convergence of $\overline{S}_{\theta_0}^{(m)}$.

Let $\Pi(s|\mathcal{S})$ denote the (componentwise) projection of a vector $s \in \bigoplus_{k=1}^d L^2(P_0)$ on a subspace $\mathcal{S} \subset \bigoplus_{k=1}^d L^2(P_0)$. Theorem A.4.2 (A) from BKRW directly yields the following result.

Lemma 2 *Assume that the conditions of Theorem 1 hold true. When $m \rightarrow \infty$,*

$$\overline{S}_{\theta_0}^{(m)} = a_1^{(m)}(X^{(1)})g_1 + a_2^{(m)}(X^{(2)})g_2 \longrightarrow \overline{S}_{\theta_0} = \Pi(S_{\theta_0}|\mathcal{T}^\perp) = \Pi(S_{\theta_0}|\overline{H_1 + H_2})$$

in $\bigoplus_{k=1}^d L^2(P_0)$, where $g_i = g(Z, \theta_0)$, $i = 1, 2$.

Let us point out that even if Lemma 2 guarantees the convergence of the iterations $\overline{S}_{\theta_0}^{(m)}$, it is not necessarily true that the sequences $a_1^{(m)}(X^{(1)})g_1$ and $a_2^{(m)}(X^{(2)})g_2$ converge. Sufficient mild conditions are provided in Theorem A.4.2 (C) of BKRW, that are

$$\overline{S}_{\theta_0} = \Pi(S_{\theta_0}|\mathcal{T}^\perp) = a_1^*(X^{(1)}) \cdot g_1 + a_2^*(X^{(2)}) \cdot g_2 \in \mathcal{T}_1^\perp + \mathcal{T}_2^\perp \quad (18)$$

with $a_1^*(X^{(1)}) \cdot g_1 \in \mathcal{T}_1^\perp \cap (\mathcal{T}_1^\perp \cap \mathcal{T}_2^\perp)^\perp \subset \mathcal{T}_1^\perp$. Moreover, by Proposition A.4.1 of BKRW, condition (18) is equivalent with the existence of a solution $a_1^*g_1$ and $a_2^*g_2$ for the system

$$\begin{cases} a_1^*(X^{(1)}) g_1 = \rho_1 - E[a_2^*(X^{(2)}) g_2 g'_1 | X^{(1)}] V^-(g_1 | X^{(1)}) g_1 \\ a_2^*(X^{(2)}) g_2 = \rho_2 - E[a_1^*(X^{(1)}) g_1 g'_2 | X^{(2)}] V^-(g_2 | X^{(2)}) g_2, \end{cases} \quad (19)$$

where

$$\begin{aligned}
\rho_i = \rho_i(Z, \theta_0) &:= \Pi(S_{\theta_0}|\mathcal{T}_i^\perp) = E(S_{\theta_0}g'_i | X^{(i)}) V^-(g_i | X^{(i)}) g_i \\
&= -E(\partial_\theta g'_i | X^{(i)}) V^-(g_i | X^{(i)}) g_i.
\end{aligned}$$

(A careful inspection of the proof of Proposition A.4.1 of BKRW shows that condition $H_1 + H_2 = \mathcal{T}_1^\perp + \mathcal{T}_2^\perp$ is a closed subspace is not necessary for deriving that result, since what is really used in their proof is the relation $H_1^\perp \cap H_2^\perp = (H_1 + H_2)^\perp$). If in addition the system (19) has a unique solution, the backfitting algorithm above is nothing but a convergent iterative procedure for finding it.

In applications, a convenient way to check uniqueness is to prove a contraction property. This is the case for instance if $\mathcal{T}_1^\perp \cap \mathcal{T}_2^\perp = \{0\}$, which in our framework holds if

$$E(g_1 g_2' \mid X^{(1)}, X^{(2)}) = 0$$

(in the sequential case, this can be achieved by writing the initial system in an equivalent form satisfying the orthogonal condition above; see subsection 4.1).

In the general case where $\mathcal{T}_1^\perp \cap \mathcal{T}_2^\perp \neq \{0\}$ the system (19) rewritten as in Proposition A.4.1 of BKRW under the form

$$\begin{cases} h_1^* = \Pi(S_{\theta_0} - h_2^* | \mathcal{T}_1^\perp) \\ h_2^* = \Pi(S_{\theta_0} - h_1^* | \mathcal{T}_2^\perp), \end{cases}$$

does not necessarily have the contraction property. In our problem $h_1^* = a_1^* g_1$ and $h_2^* = a_2^* g_2$ with g_1 and g_2 given. Hence it suffices to check a contraction property for $a_1^* g_1$ and $a_2^* g_2$ or some given transformations of them. We will see in subsection 4.2 that in the regression-like models with missing data framework, see Robins, Rotnitzky, Zhao (1994), the equations (19) lead to a contraction property for some given transformations of $a_1^* g_1$ and $a_2^* g_2$.

The “backfitting” algorithm we proposed above involves θ_0 that is unknown. In practice one can use the following steps: (i) build $\tilde{\theta}_n$ a \sqrt{n} -consistent estimator of θ_0 , for instance the *smooth minimum distance estimator (SMD)* like in Lavergne and Patilea (2008); (ii) estimate nonparametrically $a_1^{(m^*)}$ and $a_2^{(m^*)}$ the solution of the “backfitting” algorithm obtained after, say, m^* iterations using $\tilde{\theta}_n$ instead of θ_0 ; and (iii) construct an efficient (classical GMM or SMD) estimator $\hat{\theta}^{(m^*)}$ based on the approximate efficient score equations $E(\hat{S}_\theta) = 0$, where

$$\hat{S}_\theta = \hat{a}_1^{(m^*)}(X^{(1)}, \tilde{\theta}_n) g_1(Z, \theta) + \hat{a}_2^{(m^*)}(X^{(2)}, \tilde{\theta}_n) g_2(Z, \theta),$$

and $\hat{a}_i^{(m^*)}(X^{(i)}, \tilde{\theta}_n)$ are nonparametric estimates of $a_i^{(m^*)}(X^{(i)}, \theta_0)$, $i = 1, 2$.

4 Applications

In this section we illustrate the utility of our theoretical results for two general classes of models: sequential (nested) conditional models and regression-like models with missing

data. The general results in sections 2 and 3 above allow us: (a) to complete a semiparametric efficiency bound result of Chamberlain (1992b); and (b) to generalize the mean regression with missing data setting of Robins, Rotnitzky and Zhao (1994) and Tan (2011) to more general moment conditions, which includes for example quantile regressions.

4.1 Sequential conditional moments

Important cases where equations (19) have an explicit solution are the cases where $\sigma(X^{(1)}) \subset \sigma(X^{(2)})$ holds true. In the case $J = 2$, the model $E(g_j(Z, \theta) | X^{(j)}) = 0$, $j = 1, 2$, defined in (1) can be equivalently written under the form

$$\begin{cases} E(\tilde{g}_1(Z, \theta) | X^{(1)}) = 0 \\ E(g_2(Z, \theta) | X^{(2)}) = 0, \end{cases} \quad (20)$$

where

$$\tilde{g}_1(Z, \theta) = g_1(Z, \theta) - E(g_1(Z, \theta_0) | X^{(1)}) - V^{-1}(g_2(Z, \theta_0) | X^{(2)}) g_2(Z, \theta).$$

Here we suppose that $V(g_1(Z, \theta_0) | X^{(1)})$ and $V(g_2(Z, \theta_0) | X^{(2)})$ are invertible and this guarantees that θ_0 is also identified by the equations (20). Recall that g_i is a short notation for $g_i(Z, \theta_0)$ and similarly let \tilde{g}_i replace $\tilde{g}_i(Z, \theta_0)$.

Notice that \tilde{g}_1 is the residual of the projection of g_1 on g_2 with respect to $\sigma(X^{(2)})$ and $E(\tilde{g}_1 | X^{(2)}) = 0$. Let $\tilde{\mathcal{T}}_1$ be the tangent space of the model defined by the first equation in (20). By the definition of \tilde{g}_1 , it is quite clear that condition $\tilde{\mathcal{T}}_1^\perp \cap \mathcal{T}_2^\perp = \{0\}$ holds true. Next, multiplying the i th equation in (19) by g_i , taking conditional expectation given $X^{(i)}$ and finally multiplying by $V^{-1}(g_i | X^{(i)})$, $i = 1, 2$, the system (19) corresponding to model (20) becomes

$$\begin{cases} \tilde{a}_1^*(X^{(1)}) = -E(\partial_\theta \tilde{g}_1' | X^{(1)}) V^{-1}(\tilde{g}_1 | X^{(1)}) \\ \quad - E(\tilde{a}_2^*(X^{(2)}) \cdot g_2 \tilde{g}_1' | X^{(1)}) V^{-1}(\tilde{g}_1 | X^{(1)}) \\ \tilde{a}_2^*(X^{(2)}) = -E(\partial_\theta g_2' | X^{(2)}) V^{-1}(g_2 | X^{(2)}) \\ \quad - E(\tilde{a}_1^*(X^{(1)}) \cdot \tilde{g}_1 g_2' | X^{(2)}) V^{-1}(g_2 | X^{(2)}). \end{cases} \quad (21)$$

Since by definition $E(\tilde{a}_2^*(X^{(2)}) g_2 \tilde{g}_1' | X^{(1)}) = E[\tilde{a}_2^*(X^{(2)}) E(g_2 \tilde{g}_1' | X^{(2)}) | X^{(1)}] = 0$ and $E(\tilde{a}_1^*(X^{(1)}) \tilde{g}_1 g_2' | X^{(2)}) = \tilde{a}_1^*(X^{(1)}) E(\tilde{g}_1 g_2' | X^{(2)}) = 0$ we obtain

$$\begin{cases} \tilde{a}_1^*(X^{(1)}) = -E(\partial_\theta \tilde{g}_1' | X^{(1)}) V^{-1}(\tilde{g}_1 | X^{(1)}) \\ \tilde{a}_2^*(X^{(2)}) = -E(\partial_\theta g_2' | X^{(2)}) V^{-1}(g_2 | X^{(2)}). \end{cases} \quad (22)$$

($E(\partial_\theta \tilde{g}_i')$ denotes the transposed of the matrix $E(\partial_\theta \tilde{g}_i)$.) The efficient score \bar{S}_{θ_0} can then be written as

$$\begin{aligned} \bar{S}_{\theta_0} &= \tilde{a}_1^*(X) \cdot \tilde{g}_1 + \tilde{a}_2^*(X) \cdot g_2 \\ &= -E(\partial_\theta \tilde{g}_1' | X^{(1)}) V^{-1}(\tilde{g}_1 | X^{(1)}) \tilde{g}_1 - E(\partial_\theta g_2' | X^{(2)}) V^{-1}(g_2 | X^{(2)}) g_2. \end{aligned}$$

In the particular case where $X^{(1)} = X^{(2)} = X$,

$$\begin{aligned}
\overline{S}_{\theta_0} &= \tilde{a}_1^*(X) \cdot \tilde{g}_1 + \tilde{a}_2^*(X) \cdot g_2 \\
&= -E(\partial_\theta \tilde{g}_1' | X) V^{-1}(\tilde{g}_1 | X^{(1)}) \tilde{g}_1 - E(\partial_\theta g_2' | X) V^{-1}(g_2 | X) g_2 \\
&= \begin{pmatrix} -E(\partial_\theta \tilde{g}_1' | X) \\ -E(\partial_\theta g_2' | X) \end{pmatrix}' V^{-1} \left(\begin{pmatrix} \tilde{g}_1 \\ g_2 \end{pmatrix} | X \right) \begin{pmatrix} \tilde{g}_1 \\ g_2 \end{pmatrix} \\
&= -E(\partial_\theta g' C'(X) | X) V^{-1}(C(X)g | X) C(X)g \\
&= -E(\partial_\theta g' | X) V^{-1}(g | X)g,
\end{aligned}$$

where $g' = (g_1' g_2')$ and

$$C(X) = \begin{pmatrix} I & -E(g_1 g_2' | X) V^{-1}(g_2 | X) \\ 0 & I \end{pmatrix}$$

is a nonsingular random matrix. This expression of the efficient score directly yields the efficiency bound derived in Chamberlain (1987).

Another important particular case of formulae (22) is provided by models defined by sequential conditional moments; see Chamberlain (1992b), Ai and Chen (2009). Taking $X^{(1)} = X_1$ and $X^{(2)} = (X_1', X_2')'$, one obtains

$$\begin{aligned}
\overline{S}_{\theta_0} &= \tilde{a}_1^*(X) \cdot \tilde{g}_1 + \tilde{a}_2^*(X) \cdot g_2 \\
&= -E(\partial_\theta \tilde{g}_1' | X_1) V^{-1}(\tilde{g}_1 | X_1) \tilde{g}_1 - E(\partial_\theta g_2' | X_1, X_2) V^{-1}(g_2 | X_1, X_2) g_2.
\end{aligned}$$

Let us point that Chamberlain (1992b) only proves this result for discrete distributions and Ai and Chen (2009) obtain the result in a more general framework (allowing for unknown infinite dimensional parameters in the equations defining the model) but under slightly more restrictive assumptions than in our setting.¹

4.2 Regression-like models with missing data

Consider now a regression-like model defined by the equations

$$E[\rho(Y, X^*, \alpha) | X^*] = 0, \quad (23)$$

where $\rho(\cdot, \cdot, \cdot)$ is some measurable vector-valued function, α is a (finite-dimension) vector of parameters, and the vector $(Y', X^{*'}) = (Y', X', V')$ is not always completely observed.

¹Ai and Chen (2009) implicitly require that the class \mathcal{G} appearing in their Assumption A in the Mathematical Appendix is the same for each value of their model parameter α . This variation independent parametrization assumption represents an additional restriction that is unnecessary in our approach. See also van der Laan and Robins (2003), page 18, for some lucid comments on the existence of a variation independent parametrization.

We also assume that a non-missing indicator δ and some other variable V^0 are always observed. In the following examples we consider two random missingness mechanisms considered respectively by Tan (2011) and Robins, Rotnitzky and Zhao (1994).

Example 2 (i) The vector Y is observed iff $\delta = 1$;

(ii) The vector $W = \begin{pmatrix} X^* \\ V^0 \end{pmatrix}$ is always observed and we have

$$P(\delta = 1 \mid Y, W) = P(\delta = 1 \mid W) = \pi(W). \quad (24)$$

Example 3 (i) Let $X^* = \begin{pmatrix} X \\ V \end{pmatrix}$ where X is observed iff $\delta = 1$;

(ii) The vector $W = \begin{pmatrix} Y \\ V \\ V^0 \end{pmatrix}$ is always observed and we have

$$P(\delta = 1 \mid X, W) = P(\delta = 1 \mid W) = \pi(W). \quad (25)$$

Let α_0 be the true value of the parameter identified by the model (23). The equation (23) and each of (24) or (25) imply

$$E \left[\frac{\delta}{\pi(W)} \rho(Y, X^*, \alpha_0) \mid X^* \right] = 0. \quad (26)$$

We can consider this equation at the observational level even for missing X^* , since for missing values of X^* we have $\delta = 0$ which renders the equation noninformative. Note also that (24) and (25) can be written under the unified form

$$P(\delta = 1 \mid Y, X^*, W) = \pi(W).$$

Therefore, at the observational level, with any of the two examples we obtain a model like

$$\begin{cases} E \left[\frac{\delta}{\pi(W)} \rho(Y, X^*, \alpha_0) \mid X^* \right] = 0 \\ E \left[\frac{\delta}{\pi(W)} - 1 \mid W \right] = 0. \end{cases} \quad (27)$$

Moreover, like in Graham (2011, footnote 8, page 442), it can be shown that at the observational level, a model given by equation (23) and any of the missing data mechanism described in Example 2 or Example 3 is equivalent to the model defined by (27).

With our notation, Z is the vector built as the union of all the variables contained in Y, X^*, W and $\delta, \theta = \alpha, g_1(Z, \theta) = \{\delta/\pi(W)\}\rho(Y, X^*, \alpha), g_2(Z, \theta) = \{\delta/\pi(W)\} - 1,$

$X^{(1)} = X^*$ and $X^{(2)} = W$. Let ρ be a short for $\rho(Y, X^*, \alpha_0)$. Then the functions a_1^* and a_2^* defining the efficient score are given by the following equations obtained (see also equations (21)) from equations (19) :

$$\begin{aligned}
a_1^*(X^*) &= a_1^*(X^{(1)}) \\
&= -E(\partial_{\alpha}\rho' \mid X^*) E^{-1}\left(\frac{1}{\pi(W)} \rho \rho' \mid X^*\right) \\
&\quad + E\left\{E[a_1^*(X^*) \rho \mid W] \frac{1 - \pi(W)}{\pi(W)} \rho' \mid X^*\right\} E^{-1}\left(\frac{1}{\pi(W)} \rho \rho' \mid X^*\right); \\
a_2^*(W) &= a_2^*(X^{(2)}) \\
&= E\left[a_1^*(X^*) \rho \frac{\delta}{\pi(W)} \left(\frac{\delta}{\pi(W)} - 1\right) \mid W\right] E^{-1}\left[\left(\frac{\delta}{\pi(W)} - 1\right)^2 \mid W\right] \\
&= -E[a_1^*(X^*) \rho \mid W].
\end{aligned}$$

In the particular case where $\rho = \rho(Y, X^*, \alpha_0) = Y - g(X^*, \alpha_0)$ and the selection probability $\pi(W)$ is known, these are exactly the equations obtained in Robins, Rotnitzky and Zhao (1994). They showed that for the regression case, the equation for a_1^* corresponds to a contraction (see the proof of their Proposition 4.2). In subsection 5.3 in the Appendix we show that such a contraction property holds for a more general ρ . Hence we could include in our framework further interesting examples, *e.g.* quantile regressions. The contraction property allows to solve the equations in $a_1^*(X^*)$ and $a_2^*(W)$ by successive approximations.

Let us consider the extended framework where the selection probability is known up to an unknown finite dimension parameter γ_0 , that is

$$P(\delta = 1 \mid W) = \pi(W, \gamma_0),$$

(see also Robins, Rotnitzky and Zhao (1994), equation (18)). In subsection 5.4 in the Appendix we show that the efficiency score for α_0 has the same expression regardless the selection probability function π is given or depends on the unknown parameter γ_0 . Thus, we extend a result of Robins, Rotnitzky and Zhao (1994), see also Tan (2011), obtained in the particular case of mean regressions.

Let us close this section with a remark. Robins, Rotnitzky and Zhao (1994) considered the case where missingness arises only in covariables X^* (that is also the case considered in our Example 3) and derived the efficient score equations. Tan (2011) obtained formally the same equations with missing regressors *and* missing responses (the case corresponding to our Example 2) using the corresponding definition of W . However, there is an important difference between the Examples 2 and 3. In the possibly missing responses case we have $\sigma(X^*) \subset \sigma(W)$, so that Example 2 falls in the sequential conditional moments

framework where the solutions for a_1^* and a_2^* are explicit. Such explicit solutions are *no longer* available in the framework considered by Robins, Rotnitzky and Zhao (1994) and in our Example 3.

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5 Appendix

5.1 Additional proofs

Proof of Lemma 1. By definition (van der Vaart (1998), pp 363), there exists a continuous linear map $\dot{\psi} : L^2(P_0) \rightarrow \mathbb{R}^d$ such that for any $g \in \dot{\mathcal{P}}_{P_0} \subset L^2(P_0)$ and a submodel $(-\varepsilon, \varepsilon) \ni t \mapsto P_t$ with score function g ,

$$\frac{\psi(P_t) - \psi(P_0)}{t} \xrightarrow[t \rightarrow 0]{} \dot{\psi}(g).$$

By the Riesz representation theorem, there exists a unique d -dimension vector-valued function having the components in $L^2(P_0)$ such that $\dot{\psi}(h) = E_{P_0}(\bar{\psi}h)$ for every $h \in L^2(P_0)$. In particular,

$$\dot{\psi}(g) = E_{P_0}(\bar{\psi}g) = \int \bar{\psi}g dP_0, \quad \forall g \in \dot{\mathcal{P}}_{P_0} \subset L^2(P_0).$$

Let $\tilde{\bar{\psi}}$ and $\tilde{\bar{\psi}}_k$ denote the elements of $[L^2(P_0)]^d$ obtained by componentwise projections of $\bar{\psi}$ on the tangent spaces $\mathcal{T} \subset L^2(P_0)$ and $\mathcal{T}_k \subset L^2(P_0)$, respectively. The Fisher information matrices on $\theta_0 = \psi(P_0)$ in the models $\mathcal{P}, \mathcal{P}_k$ at P_0 are then defined by

$$I_{\theta_0}^{-1}(\mathcal{P}) = V_{P_0}(\tilde{\bar{\psi}}) = E_{P_0}(\tilde{\bar{\psi}}\tilde{\bar{\psi}}'), \quad I_{\theta_0}^{-1}(\mathcal{P}_k) = V_{P_0}(\tilde{\bar{\psi}}_k), \quad k \in \mathbb{N}^*.$$

From

$$\mathcal{P}_1 \supset \mathcal{P}_2 \supset \dots \supset \mathcal{P}_k \supset \mathcal{P}_{k+1} \supset \dots \supset \bigcap_{k=1}^{\infty} \mathcal{P}_k \supset \mathcal{P}$$

we deduce that

$$\dot{\mathcal{P}}_1 \supset \dot{\mathcal{P}}_2 \supset \dots \supset \dot{\mathcal{P}}_k \supset \dot{\mathcal{P}}_{k+1} \supset \dots \supset \bigcap_{k=1}^{\infty} \dot{\mathcal{P}}_k \supset \dot{\mathcal{P}},$$

and

$$\mathcal{T}_1 \supset \mathcal{T}_2 \supset \dots \supset \mathcal{T}_k \supset \mathcal{T}_{k+1} \supset \dots \supset \bigcap_{k=1}^{\infty} \mathcal{T}_k = \mathcal{T},$$

where the last equality is due to (4). By Lemma 4.5 of Hansen and Sargent (1991),

$$\lim_{k \rightarrow \infty} I_{\theta_0}^{-1}(\mathcal{P}_k) = \lim_{k \rightarrow \infty} V_{P_0}(\prod (\bar{\psi}|\mathcal{T}_k)) = V_{P_0}(\prod (\bar{\psi}|\mathcal{T})) = V_{P_0}(\tilde{\bar{\psi}}) = I_{\theta_0}^{-1}(\mathcal{P}).$$

■

5.2 Assumptions

For a subset $A \subset \text{supp}Z$, we use the following notations : $g_{i,A} = g_i(Z, \theta_0)\mathbf{I}_{\{Z \in A\}}$, $i = 1, 2$, and

$$b_i = g_{i,A} - E(g_{i,A} \mid X^{(1)}, X^{(2)}) - E^{-1}(g_{j,A} \mid X^{(1)}, X^{(2)}) g_{j,A}, \quad (28)$$

$(i, j) \in \{(1, 2), (2, 1)\}$ where $E^{-1}(g_{j,A} \mid X^{(1)}, X^{(2)})$ stands for the inverse of the matrix $E(g_{j,A} \mid X^{(1)}, X^{(2)})$ that is supposed to exist.

Assumption T

There exist a subset $A \subset \text{supp}Z$ such that for $i = 1, 2$, $g_{i,A}$ is a bounded function and

1. $E(g_{i,A} \mid X^{(1)}, X^{(2)})$ is invertible and $\|E^{-1}(g_{i,A} \mid X^{(1)}, X^{(2)})\|_\infty < \infty$;
2. $\|E^{-1}(b_i \mid X^{(i)})\|_\infty < \infty$ with b_i defined in (28).

Remark 3 Under Assumption T and for any $\alpha > 0$, by the definition of b_i , for $(i, j) \in \{(1, 2), (2, 1)\}$,

$$E(g_i(Z, \theta_0) \alpha b'_j \mid X^{(1)}, X^{(2)}) = E(g_{i,A} \alpha b'_j \mid X^{(1)}, X^{(2)}) = 0,$$

and, for $i = 1, 2$,

$$E(g_{i,A} \alpha b'_i \mid X^{(i)}) = \alpha E(b_i \mid X^{(i)}).$$

Therefore, in the proof of Theorem 1, up to a suitable scaling factor, we can choose b_1 and b_2 such that conditions (2.2) are satisfied.

Assumption SP

1. The models \mathcal{P} defined by (1) and \mathcal{P}_k defined by (6), with $k \in \mathbb{N}^*$, can be written in the semiparametric form

$$\mathcal{P} = \{P_{\theta, \eta} : \theta \in \Theta, \eta \in H\}, \quad \mathcal{P}_k = \{P_{\theta, \eta} : \theta \in \Theta, \eta \in H_k\}, \quad k \in \mathbb{N}^*,$$

and satisfy the assumptions of Lemma 25.25 (page 369) of van der Vaart (1998).

2. The Fisher information matrices I_{θ_0} and $I_{\theta_0}^{(k)}$ on θ_0 in models \mathcal{P} and \mathcal{P}_k respectively, for any $k \in \mathbb{N}^*$, are well defined and nonsingular.

To guarantee Assumption SP.2 it suffices to suppose that for any $1 \leq j \leq J$: (i) $\|V(g_j(Z, \theta_0) \mid X^{(j)})\|_\infty < \infty$; (ii) the maps $\theta \mapsto E(g_j(Z, \theta_0) \mid X^{(j)} = x^{(j)})$ are differentiable for $P_{X^{(j)}} - \text{almost all } x^{(j)}$; and (iii) the information matrix

$$E \left\{ E \left[(\partial_{\theta'} g_j(Z, \theta_0))' \mid X^{(j)} \right] V^{-} [g_j(Z, \theta_0) \mid X^{(j)}] E \left[\partial_{\theta'} g_j(Z, \theta_0 \mid X^{(j)}) \right] \right\}$$

is non singular.

A consequence of Assumption SP (see Lemma 25.25 of van der Vaart (1998)) is that the parameter defined by $\psi(P_{\theta, \eta}) = \theta$ is differentiable at $P_0 = P_{\theta_0, \eta_0}$ with respect to the tangent space $\mathcal{T} = \mathcal{T}(\mathcal{P}, P_0)$. It also ensures that the tangent space \mathcal{T} can be written as the sum of the finite dimensional subspace spanned by the components of the parametric score S_{θ_0} and the tangent space \mathcal{T}' corresponding to the nonparametric part $\mathcal{P}' = \{P_{\theta_0, \eta} : \eta \in H\}$ of the model \mathcal{P} :

$$\mathcal{T} = \text{lin} S_{\theta_0} + \mathcal{T}'.$$

Note that this assumption does not necessarily mean that the parameters θ and η are completely separated. In fact θ and η are connected since the functional parameter η can have θ among its arguments. Assumption SP only means that when considering the density of $P_{\theta, \eta}$ with respect to a dominating measure μ we could write it under the form

$$f(\cdot, \theta, \eta(v(\cdot, \theta))),$$

with f and v having a known form, where $f(\cdot, \theta_0, \eta(v(\cdot, \theta_0)))$ and $f(\cdot, \theta, \eta_0(v(\cdot, \theta)))$ belong to the model \mathcal{P} for every $\theta \in \Theta$ and $\eta \in H$. For example, in the conditional mean setting with one conditioning vector

$$E[Y - m(X, \theta) \mid X] = 0,$$

we can take H as the set of zero conditional mean densities of $Z = (Y', X')'$, i.e.

$$H = \left\{ p(y, x) \cdot \gamma(x) : p \geq 0, \gamma \geq 0, \int p(y, x) dy = 1, \int yp(y, x) dy = 0, \forall x, \int \gamma(x) dy = 1 \right\}$$

and $v(y, x, \theta) = (y - m(x, \theta), x)$, so that

$$\eta(v(z, \theta)) = \eta(y - m(x, \theta), x) = p(y - m(x, \theta), x) \cdot \gamma(x)$$

and

$$f(z, \theta, \eta(v(z, \theta))) = \eta(v(z, \theta)).$$

In the proof of Theorem 1 we identify the density $f(\cdot, \theta, \eta(v(\cdot, \theta)))$ with the infinite dimensional nuisance parameter η which is itself a density.

5.3 Contraction property in regression-like models with missing data

With the same notation of subsection 4.2, we shall prove that the equation

$$\begin{aligned} a_1^*(X^*) &= E \left\{ E[a_1^*(X^*) \rho(Z, \theta_0) \mid W] \frac{1 - \pi(W)}{\pi(W)} \rho'(Z, \theta_0) \mid X^* \right\} \\ &\times E^{-1} \left[\frac{1}{\pi(W)} \rho(Z, \theta_0) \rho'(Z, \theta_0) \mid X^* \right] \end{aligned} \quad (29)$$

has a unique solution which can be obtained by successive approximation, under the additional assumption

$$\inf_w \pi(w) = 1 - \beta > 0, \quad (30)$$

the infimum being taken over all possible values of W . For simplicity, in the reminder of this subsection we drop the arguments of the functions. Let $\tilde{\rho} = \pi^{-1/2}\rho$. Assuming that $E(\tilde{\rho} \tilde{\rho}' \mid X^*)$ is invertible, equation (29) can be equivalently written under the form

$$\begin{aligned} a_1^* \tilde{\rho} &= E \left[E(a_1^* \rho \mid W) \frac{1 - \pi}{\pi} \rho' \mid X^* \right] E^{-1} \left(\frac{1}{\pi} \rho \rho' \mid X^* \right) \tilde{\rho} \\ &= E[E(a_1^* \tilde{\rho} \mid W) (1 - \pi) \tilde{\rho}' \mid X^*] E^{-1}(\tilde{\rho} \tilde{\rho}' \mid X^*) \tilde{\rho} \\ &=: \tilde{T}(a_1^* \tilde{\rho}). \end{aligned}$$

We will show that the map \tilde{T} is a contraction. Before that, let us state a Cauchy-Schwarz inequality for matrix valued random variables, a version of an inequality in Lavergne (2008): let \mathbb{E} denote the conditional expectation given an arbitrary σ -field, let $A \in \mathbb{R}^n \times \mathbb{R}^p$ and $B \in \mathbb{R}^n \times \mathbb{R}^q$ be random matrices such that $\mathbb{E}(\text{tr}(A'A)), \mathbb{E}(\text{tr}(B'B)) < \infty$ and $\mathbb{E}(A'A)$ is non-singular. Then $\mathbb{E}(B'B) - \mathbb{E}(B'A)\mathbb{E}^{-1}(A'A)\mathbb{E}(A'B)$ is positive semi-definite, with equality iff $B = A\mathbb{E}^{-1}(A'A)\mathbb{E}(A'B)$.² We also use the following notation: for any symmetric matrices B_1, B_2 , $B_1 \gg B_2$ means $B_1 - B_2$ is positive semi-definite. Let

²Like in Lavergne (2008), let $\Lambda = \mathbb{E}^{-1}(A'A)\mathbb{E}(A'B)$. Then

$$\mathbb{E}[(B - A\Lambda)'(B - A\Lambda)] = \mathbb{E}(B'B) - \mathbb{E}(B'A)\mathbb{E}^{-1}(A'A)\mathbb{E}(A'B)$$

is clearly positive semi-definite, and is zero iff $B = A\Lambda$.

us write

$$\begin{aligned}
E[\tilde{T}(a_1^* \tilde{\rho}) \tilde{T}'(a_1^* \tilde{\rho})] &= E \left\{ [E(a_1^* \tilde{\rho} | W) (1 - \pi) \tilde{\rho}' | X^*] E^{-1}(\tilde{\rho} \tilde{\rho}' | X^*) \tilde{\rho} \right. \\
&\quad \times \tilde{\rho}' E^{-1}(\tilde{\rho} \tilde{\rho}' | X^*) \{[E(a_1^* \tilde{\rho} | W) (1 - \pi) \tilde{\rho}' | X^*]\}' \} \\
&= E \left\{ [E(a_1^* \tilde{\rho} | W) (1 - \pi) \tilde{\rho}' | X^*] E^{-1}(\tilde{\rho} \tilde{\rho}' | X^*) \right. \\
&\quad \times \{[E(a_1^* \tilde{\rho} | W) (1 - \pi) \tilde{\rho}' | X^*]\}' \} \\
(\text{Cauchy-Schwarz}) &\ll E \left\{ E[E(a_1^* \tilde{\rho} | W) (1 - \pi)^2 E(\tilde{\rho}' a_1^{*'} | W) | X^*] \right\} \\
&= E[E(a_1^* \tilde{\rho} | W) (1 - \pi)^2 E(\tilde{\rho}' a_1^{*'} | W)] \\
(\text{Cauchy-Schwarz}) &\ll E[(1 - \pi)^2 (a_1^* \tilde{\rho}) (a_1^* \tilde{\rho})']
\end{aligned}$$

This implies

$$\begin{aligned}
\|\tilde{T}(a_1^* \tilde{\rho})\|_{L^2}^2 &= E \left\{ \text{tr} \left[\tilde{T}'(a_1^* \tilde{\rho}) \tilde{T}(a_1^* \tilde{\rho}) \right] \right\} = \text{tr} \left\{ E \left[\tilde{T}(a_1^* \tilde{\rho}) \tilde{T}'(a_1^* \tilde{\rho}) \right] \right\} \\
&\leq \sup_w [1 - \pi(w)] \|a_1^* \tilde{\rho}\|_{L^2}^2 \leq \beta \|a_1^* \tilde{\rho}\|_{L^2}^2,
\end{aligned}$$

where $\beta = \sup_w [1 - \pi(w)] = 1 - \inf_w \pi(w) < 1$ by assumption (30). Deduce that \tilde{T} is a contracting map.

5.4 Efficient score with parametric selection probability in regression-like models with missing data

Let $X^{(1)} = X^*$, $X^{(2)} = W$ and the parameter vector $\theta = (\alpha', \gamma')'$. Moreover, let

$$\begin{aligned}
g_1(Z, \theta) &= \frac{\delta}{\pi(W, \gamma)} \rho(Y, X^*, \alpha), \quad g_2(Z, \theta) = \frac{\delta}{\pi(W, \gamma)} - 1, \\
\bar{S}_\theta &= \bar{a}_1(X^*) g_1(Z, \theta) + \bar{a}_2(W) g_2(Z, \theta) = \begin{pmatrix} \bar{S}_\alpha \\ \bar{S}_\gamma \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\bar{a}_1(X^*) &= \bar{a}_1(X^{(1)}) \\
&= \begin{pmatrix} -E[E(\pi^{-1}(W, \gamma_0) \delta | X^*, W) \partial_\alpha \rho' | X^*] E^{-1}(\pi^{-1}(W, \gamma_0) \rho \rho' | X^*) \\ 0 \end{pmatrix} \\
&+ E \left\{ E[\bar{a}_1(X^*) \rho | W] (\pi^{-1}(W, \gamma_0) - 1) \rho' | X^* \right\} E^{-1}(\pi^{-1}(W, \gamma_0) \rho \rho' | X^*).
\end{aligned}$$

If we partition $\bar{a}_1(X^*)$ in $\bar{a}_1(X^*) = \begin{pmatrix} \bar{a}_{1,\alpha}(X^*) \\ \bar{a}_{1,\gamma}(X^*) \end{pmatrix}$ and we use the same short notation as previously, the preceding equations can be written as

$$\begin{aligned}\bar{a}_{1,\alpha}(X^*) &= -E \left(E \left(\frac{\delta}{\pi} \mid X^*, W \right) \partial_{\alpha} \rho' \mid X^* \right) E^{-1} \left(\frac{1}{\pi} \rho \rho' \mid X^* \right) \\ &\quad + E \left\{ E [\bar{a}_{1,\alpha}(X^*) \rho \mid W] \left(\frac{1}{\pi} - 1 \right) \rho' \mid X^* \right\} \\ &\quad \times E^{-1} \left(\frac{1}{\pi} \rho \rho' \mid X^* \right), \\ \bar{a}_{1,\gamma}(X^*) &= E \left\{ E [\bar{a}_{1,\gamma}(X^*) \rho \mid W] \left(\frac{1}{\pi} - 1 \right) \rho' \mid X^* \right\} \\ &\quad \times E^{-1} \left(\frac{1}{\pi} \rho \rho' \mid X^* \right),\end{aligned}$$

with the obvious solution $\bar{a}_{1,\gamma} \equiv 0$ for the subvector of \bar{a}_1 corresponding to γ (possibly not the unique solution, but any solution yields the same efficient score \bar{S}_θ). Similar calculations can be done for $\bar{a}_2(W)$:

$$\begin{aligned}\bar{a}_2(W) &= \bar{a}_2(X^{(2)}) \\ &= \begin{pmatrix} 0 \\ \frac{1}{\pi} \partial_{\gamma} \pi \end{pmatrix} \frac{\pi}{1 - \pi} - E [\bar{a}_1(X^*) \rho \mid W],\end{aligned}$$

which gives, for $\bar{a}_2(W) = \begin{pmatrix} \bar{a}_{2,\alpha}(W) \\ \bar{a}_{2,\gamma}(W) \end{pmatrix}$,

$$\begin{aligned}\bar{a}_{2,\alpha}(W) &= -E [\bar{a}_{1,\alpha}(X^*) \rho \mid W] \\ \bar{a}_{2,\gamma}(W) &= \frac{1}{1 - \pi} \partial_{\gamma} \pi - E [\bar{a}_{1,\gamma}(X^*) \rho \mid W] = \frac{1}{1 - \pi} \partial_{\gamma} \pi.\end{aligned}$$

Therefore,

$$\bar{S}_\theta = \bar{a}_1(X^*) g_1 + \bar{a}_2(W) g_2 = \begin{pmatrix} \bar{S}_\alpha \\ \bar{S}_\gamma \end{pmatrix} = \begin{pmatrix} \bar{a}_{1,\alpha}(X^*) g_1 + \bar{a}_{2,\alpha}(W) g_2 \\ \bar{a}_{2,\gamma}(W) g_2 \end{pmatrix},$$

where

$$\begin{aligned}\bar{a}_{1,\alpha}(X^*) &= -E(\partial_\alpha \rho' \mid X^*) E^{-1} \left(\frac{1}{\pi} \rho \rho' \mid X^* \right) \\ &\quad + E \left\{ E[\bar{a}_{1,\alpha}(X^*) \rho \mid W] \frac{1-\pi}{\pi} \rho' \mid X^* \right\} \\ &\quad \times E^{-1} \left(\frac{1}{\pi} \rho \rho' \mid X^* \right),\end{aligned}$$

$$\bar{a}_{2,\alpha}(W) = -E[\bar{a}_{1,\alpha}(X^*) \rho \mid W],$$

$$\bar{a}_{2,\gamma}(W) = \frac{\pi}{1-\pi} \partial_\gamma \pi \quad \left(= \frac{\pi(W, \gamma_0)}{1-\pi(W, \gamma_0)} \partial_\gamma \pi(W, \gamma_0) \right).$$

Now, for any $s = b(W) \cdot g_2 = b(W) \left(\frac{\delta}{\pi(W, \gamma_0)} - 1 \right) \in \mathcal{T}_2^\perp$, we have

$$\begin{aligned}E(\bar{S}_\alpha s' \mid W) &= E \left[\bar{S}_\alpha \left(\frac{\delta}{\pi(W, \gamma_0)} - 1 \right) \mid W \right] b'(W) \\ &= E \left\{ \left[\bar{a}_{1,\alpha}(X^*) \frac{\delta}{\pi} \rho + \bar{a}_{2,\alpha}(W) \left(\frac{\delta}{\pi} - 1 \right) \right] \left(\frac{\delta}{\pi} - 1 \right) \mid W \right\} b'(W) \\ &= \{E[\bar{a}_{1,\alpha}(X^*) \rho \mid W] + \bar{a}_{2,\alpha}(W)\} \left(\frac{1}{\pi} - 1 \right) b'(W) \\ &= \{E[\bar{a}_{1,\alpha}(X^*) \rho \mid W] - E[\bar{a}_{1,\alpha}(X^*) \rho \mid W]\} \left(\frac{1}{\pi} - 1 \right) b'(W) \\ &= 0,\end{aligned}$$

so that, since $\bar{S}_\gamma = \bar{a}_{2,\gamma}(W) \cdot g_2$, we obtain

$$E(\bar{S}_\alpha \bar{S}_\gamma') = E[E(\bar{S}_\alpha \bar{S}_\gamma' \mid W)] = 0.$$

This means that the efficient score S_α^* for α , equal to the residual of the (componentwise) projection of \bar{S}_α on \bar{S}_γ , coincides with \bar{S}_α ,

$$S_\alpha^* = \bar{S}_\alpha - E(\bar{S}_\alpha \bar{S}_\gamma') V^{-1}(\bar{S}_\gamma) \bar{S}_\gamma = \bar{S}_\alpha,$$

and has the same expression, as already noticed in Robins, Rotnitzky and Zhao (1994), as in the case where $\pi(W)$ is completely known :

$$\begin{aligned}S_\alpha^* &= \bar{S}_\alpha = \bar{a}_{1,\alpha}(X^*) g_1 + \bar{a}_{2,\alpha}(W) g_2 \\ &= a_1^*(X^*) g_1 + a_2^*(W) g_2.\end{aligned}$$